ON SOME PROBLEMS OF FLUID MOTION INVOLVING FREE SURFACES

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In the first part two schemes of flow about bodies by jets of finite width are considered. The fluid is assumed to be ideal. Primary attention is given to the plane case, but the possibility of considering three-dimensional formulations is also indicated. Relying on the described schemes and also taking viscosity into account in the qualitative manner, we present, in the second part, an explanation of the two following phenomena.

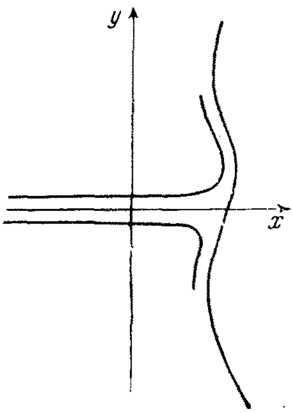

FIG. 1

A small light ball (made of cork or a ping-pong ball), located in a thin jet (of air or water directed vertically upwards), can be held stably in this jet. This phenomenon has been known for a long time, and certain toys are based on it.

Let us imagine a circular cylinder (of the length several times greater than its diameter) which is able to rotate about its own axis almost without friction. Let us place the cylinder so that its axis is horizontal, and let us direct a jet of air or water on the cylinder. We assume that the axis of the jet (at the moment of formation) is horizontal and passes below the axis of the cylinder. If the diameter of the jet is small in comparison to the diameter of the cylinder, then the rotation of the cylinder occurs in the naturally expected direction — the velocity of the lower part of the cylinder will have the same direction as that of the velocity of the jet. However, it has been shown that over a definite range of jet thicknesses and of displacements of the jet axis downwards from the cylinder axis, the lower part of the cylinder will acquire a velocity in the opposite direction.

This effect was first discovered by M.A. Gol'shtik, and who also determined experimentally the moments of the flow forces acting on the cylinder.

1. Let us begin with some formulations of problems of the motion of an ideal fluid.

A body A (finite or infinite, figure 1), which is bounded by a line Γ located to the right of the y-axis, is given. It is required to determine the fluid flow which satisfies the following conditions.

1°. At $x \to -\infty$ the desired motion becomes the translational motion of a jet $|y| \leqslant h$ parallel to the x-axis, the velocity of which is equal to unity.

2°. On the free surface the velocity is constant, i.e. (in view of 1°) the velocity is equal to unity.

3°. The motion outside the body A is a potential one, and without singularities.

Let us investigate the stability and existence of the solution. Let us consider the case in which Γ is given by the equation

$$x = x(y), \quad x(y) > 0$$

where $x(y)$ is a single-valued, twice-differentiable function.

We note that for the boundary condition $V = 1$ on the free surface the solution of the problem is unstable; namely, flows can be constructed with arbitrarily large deviations on the free surface $|V - 1| < \varepsilon$ (ϵ is arbitrarily small).

Indeed, let the lines $y = y_1(x)$ and $y = y_2(x)$ for $x < -1$ be the boundaries of the desired flow.

Let us consider a flow with the velocity equal to 1 at $x = -\infty$, bounded by the lines $y_1 = y_1(x)$ and $y_2 = y_2(x)$ for $|x - 2n| \geqslant n$ and by the lines

$$y = y_1(x) + \frac{\sqrt{n}}{2} \cos\left(\frac{\pi x}{n} + 1\right)$$

$$y = y_2(x) + \frac{\sqrt{n}}{2} \cos\left(\frac{\pi x}{n} + 1\right)$$

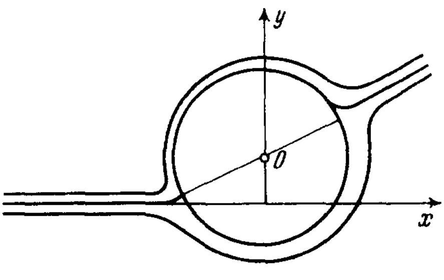

FIG. 2

It is not difficult to see that, for sufficiently large n, the constructed flow will satisfy the condition $V = 1$ on the boundary with arbitrary accuracy while its boundary will differ by an arbitrary amount from the boundary of the exact solution.

Stability will be achieved if the solution is sought in a class of domains, which for $x < -A$ satisfy the condition

$$y_1(x) + h| \leqslant Be^x \; y_2(x) - h| \leqslant Be^x \; (B = \text{const})$$

Here $y = y_1(x)$, and $y = y_2(x)$ are the boundaries of the jet.

We shall indicate a way to construct the solution. The complex potential of the desired flow $w = f(z)$ will obviously satisfy the conformal transformation of the region of flow onto the region Δ, obtained from the strip $|v| < h$ by discarding the ray

$$v = v_0,\ u > 0,\ |v_0| < h \tag{1}$$

In addition, free boundary of the flow (figure 2) must correspond to the boundary of the boundary of the strip and the ray (1) must correspond to the line Γ.

Free boundaries Γ_1 and Γ_2 must be chosen so that on Γ_1 and Γ_2

$$|f'(z)| = 1$$

Let $z = \varphi(w)$ be an inverse function of f. The problem under consideration is then reduced to the following. A function

$$F(w) = \log \varphi'(w) = a(u, v) + ib(u, v)$$

must be determined in the region Δ so, that the following conditions are satisfied.

On the straight lines $v = \pm h$ the function

$$a(u, v) = a(u, \pm h) = 1$$

On the ray $v = v_0$, $u > 0$ the function $b(u, v_0)$ should be defined in such a way, that the slope of the tangent of Γ at the point which corresponds to the point u, $u > 0$ of the ray, is equal to $b(u, v_0)$.

The quantity $b(u, v_0)$ can be given 'in an arbitrary manner' as a function of u; then, having solved the mixed boundary value problem, we can obtain classes of motion for various Γ.

Existence and uniqueness theorems can be obtained by the variational method. The method of successive approximations will be effective here: with the lower free boundary fixed we can choose the upper one, then, with the upper boundary fixed, a new lower one can be selected, etc.

The above-mentioned arguments can be extended also to the case in which A is of finite size. In contrast to the case examined previously the solution of the problem will not be uniquely determined defining the body and the jet at $x \to -\infty$. This approach will yield a set of solutions which depend on a single parameter. This parameter can be determined, for example, by assuming either the junction point of the jet behind the body or the velocity circulation about the body, known.

Let us consider an approximate method of solving the problem for small h. Let Γ be a circle of unit radius with center on the y-axis.

Under these conditions the adjacent circular arc in the vicinity of the point of division of the jet (figure 2) can be regarded to within small quantities of higher order as a straight line, tangent to Γ at the point of its intersection with the x-axis. Then, according to the momentum theorem,

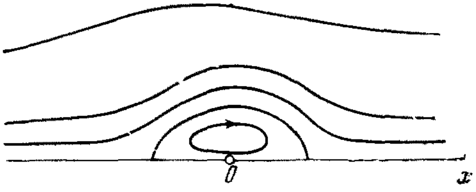

FIG. 3

$$\frac{h_2}{h_1} = \frac{1 - \sin\alpha}{1 + \sin\alpha}$$

is easily obtained for the thicknesses h_1 and h_2 of the upper and lower jet respectively.

Here α is the angle formed by the x-axis and the circle Γ.

In addition, from the geometry of the motion it follows that the flow in the vicinity of the point of recombination of the jet behind the body must be symmetric with respect to the flow in the zone of division of the jet: the flow will be symmetric with respect to some line which passes through the center of the circle.

Relying on approximate formulas of conformal transformations of narrow strips and on the condition that the velocity must be constant on a free surface, it is not difficult to see, that outside the immediate vicinity of the points of division and recombination of the jets the free surfaces of the divided jet can be taken as circles with the corresponding radii $1 + h_1$ and $1 + h_2$.

The above-mentioned approximate solution to the problem can be directly applied to the case of a body bounded by an arbitrary, sufficiently smooth curve (the curvature must satisfy the Hölder condition). The solution can be improved by taking into account the stream velocity in the thin strip in the transverse direction

$$\frac{\partial V}{\partial n} = \frac{\partial \alpha}{\partial s} = K$$

Here K is the average curvature of the boundaries of the strip.

Using an approximate theory of fluid motion between two surfaces close to each other, it is possible to construct an approximate solution to the problem of the flow around a sphere (a closed surface) by a thin jet having a cylindrical form at the point $x = -\infty$.

2. Let us consider the stability of a small ball in a vertical jet. We shall return to the plane problem of the flow around a circle of a thin jet. It is physically obvious that, if the axis of the jet passes through the center of the circle about which it flows, the point of recombination of the jet will then be found on the same diameter as the point of division of the jet. A motion in which the point of recombination is displaced, which is possible within the scheme of an ideal fluid, will not be realized since in presence of even a small amount of viscosity, the velocity loss in the jet on the longer part will be greater than on the shorter part and the jet with the greater velocity will push the jet with the lesser velocity back in the direction of the end of the diameter on which the point of division lies.

The principle so formulated can be extended also to the case in which the axis of the jet does not pass through the center of the circle. As in the symmetric case, we can assume in the first approximation that the point of recombination will be found on the same diameter as the point of division. If it is additionally considered that a thicker part of the jet over exactly the same length of run will lose less velocity than a thinner part, we then come to the conclusion that the axis of the jet will be located somewhat above the end of the diameter on which the point of division of the jet lies.

Hence, using the concept of an ideal fluid, we can make the following conclusion: if the axis of the jet intersects a contour around which it flows, at the angle α, then after flowing around the contour the axis of the jet will form an angle greater than 2α with the x-axis as $x \to +\infty$½ the jet will act on the circle with a force proportional to α in a direction perpendicular to the diameter of the circle on which the point of division of the jet lies; the force will be directed toward the axis of the jet and a small ball will be stable in the jet.

Since the thicker part of the jet occupies more than half of the circumference about which it flows, a second fact is implied: if the axis of the jet passes below the center of the circle, the circle will then rotate counter-clockwise.

3. Let us consider some schemes of fluid motion when zones of vorticity are present in the flow region. We can assume that the simplest case of such motion is the motion of an ideal fluid along the x-axis under the following conditions. The motion is potential outside some region D, and a velocity V_0 is given at infinity. In the region D the flow has constant vorticity of intensity ω.

We wish to determine the line of separation i.e. the boundary of D, in such a manner, that the flow velocity would vary continuously across the boundary.

Such a motion exists and is unique. By virtue of the similarity principle the region D is arbitrarily large for a fixed velocity V_0 and a very small ω, contracts similarly with increasing ω and its diameter tends to zero as $\omega \to \infty$.

Let us now consider the same problem for the case in which the basic motion is that of a jet of finite width h.

Let us assume that the desired solution of the problem is a region D whose area and diameter are finite as well as the curvature of its boundary (the curvature not greater than a given constant). For these conditions, if $h \leqslant h_0$, where h_0 is some constant, the solution of the problem does not exist.

Let us deduce a qualitative proof of this statement. We shall assume in addition that the boundary Γ of the region D is a convex line. Let us now assume that h is small in comparison with the diameter and with the magnitude of the inverse derivative of the curvature K of the boundary Γ.

Let us consider a point A of the line Γ which is located on the x-axis and a point B on Γ at a distance a from the point A. Let a be small, but large in comparison with h

$$a \approx \sqrt{h}$$

Then, at the point B the flow velocity in D will be small since the velocity of the flow at A is equal to zero and the derivative of the velocity is finite. On the other hand, from the condition that $a \gg h$ it follows that the velocity of the potential flow at point B will be arbitrarily close to V_0.

By the same reasoning one can be convinced of the existence and non-existence (for sufficiently small h and finite sizes of D) of motions of the following form. A region occupied by the fluid consists of the regions D and Δ; region D is bounded by the segment

oa of the x-axis, the segment ob of the y-axis and the line Γ; motion of the fluid in D is a motion with constant vorticity ω; region Δ is a strip bounded from below by the ray (a, ∞)

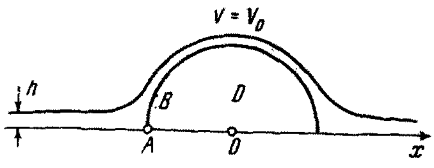

FIG. 4

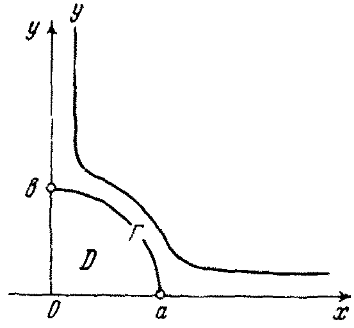

FIG. 5

of the x-axis, the ray (b, ∞) of the y-axis and the line Γ and from above by the line γ with asymptotes $y = h$ and $x = h$; the motion in Δ is potential. On the boundary Γ the velocities of both flows coincide. It would be of interest to obtain an estimate for h as a function of the dimensions of D for which a solution does or does not exist.

4. We shall return to the problem of the flow of a jet around a cylinder. In the symmetric flow of a jet of finite or infinite width around a cylinder, vortices form behind the body in the zone of recombination of the jet. We shall describe the scheme in terms of an ideal fluid, which is closest to reality. Let the center of the circle in the flow (a section of the cylinder) be located at the origin of the coordinate system and let the velocity of the jet at infinity be parallel to the x-axis. Then, a part of the flow lying above the x-axis will consist of a flow with constant vorticity ω in a region D bounded by an arc of the circle in the flow, a segment of the x-axis and an arc Γ which joins the end of the segment of the circle to the end of the segment of the x-axis; outside D the flow is potential. If the width of the jet $2h$ is infinite or large in comparison with the radius Γ of the circle in the flow, then, the dimension of D can be arbitrary and range from zero to some magnitude kr $(k < 1)$. With decreasing h the constant k will also decrease and for small h $(h \ll r)$, the limiting dimension of D will be of the order of h (just as in the flow, described earlier, interior to a coordinate angle).

If the fluid is assumed to be viscous, then the steady-state motion described above does not exist; the small vortical zone which is formed will grow and at some moment will separate from the body; from the considerations described above it can be inferred that for smaller values of h the moment of separation of the vortical zone will correspond to smaller diameters of the vortical zone.

It would be interesting to check experimentally, whether the separation will correspond to the maximum dimension of D in the simplest scheme of an ideal fluid.

Let us return to the fundamental physical problem of the flow of a jet around a cylinder when the width of the jet is commensurate with the dimensions of the body and the axis of the jet does not pass through the axis of the cylinder. In view of what has been said above, we shall take an irrotational flow as the basic motion.

Even for arbitrarily small viscosity, with the basic motion adopted at the initial moment, two vortical zones D_1 and D_2, increasing with time, would begin to form in the zone of recombination of the jet. Both vortices will separate after reaching a critical size; however, the separated vortical zone of the thin jet will be smaller than the corresponding zone of the thick jet. In the region of a thick jet (on the average) the segment of the circumference along which flow reverse with respect to the motion of the jet occurs, will be larger than the corresponding segment in the region of a thin jet. Because of friction these reverse flows will introduce additional moments. Since the moment in the zone of a thick jet will be larger than in the zone of a thin one, the resulting moment will then rotate the cylinder in a direction opposite to the motion of a thick jet.

Translated by R.D.C.